1 Trigonometrične neenakosti v trikotniku

V Hoshimo [1] zasledimo obravnavo prvih 7 neenakosti kot 'sedem svetovnih čudes'. 'Čudesa' so trigonometrične neenačbe, ki veljajo v trikotniku ABC z notranjimi koti α, β in γ. Večino izmed njih dokažemo s pomočjo analize izbočenosti kotnih funkcij. V ta namen uporabimo Jensenovo neenakost, poleg že neenakosti za konveksne,

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \geq f\left(\frac{x_1 + x_2 + x_3}{3}\right),$$

še za konkavne funkcije:

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \leq f\left(\frac{x_1 + x_2 + x_3}{3}\right).$$

Neenakost 1

$$\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$$

Dokaz 1 Ker je $\sin x$ konkavna na intervalu $[0, \pi]$ velja

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \leq \sin\left(\frac{\alpha + \beta + \gamma}{3}\right) = \sin(\pi/3) = \frac{\sqrt{3}}{2} \qquad \square$$

Neenakost 2

$$\csc\alpha + \csc\beta + \csc\gamma \geq 2\sqrt{3}$$

Dokaz 2 Funkcija $\csc x$ je konveksna na $(0, \pi)$ zato po Jansenovi neenakosti sledi

$$\csc\alpha + \csc\beta + \csc\gamma \geq 3\csc[\frac{\alpha + \beta + \gamma}{3}] = 3\csc(\pi/3) = 2\sqrt{3}. \qquad \square$$

Neenakost 3

$$1 < \cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$$

Dokaz 3 Ker je $\cos x$ konkavna na $(0, \pi/2)$, lahko Jansenovo neenakost uporabimo le za ostrokoti trikotnik. Tedaj velja

$$\cos\alpha + \cos\beta + \cos\gamma \leq 3\cos[(\alpha + \beta + \gamma)/3] = 3\cos(\pi/3) = \frac{3}{2}. \qquad \square$$

Če trikotnik ni ostrokoten, velja $\Omega = \max\{\alpha, \beta, \gamma\} \geq \pi/2$. Preoblikujemo

$$\begin{aligned}
\cos\alpha + \cos\beta + \cos\gamma &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \left(1 - 2\sin^2\frac{\gamma}{2}\right) = \\
&= 2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right) + 1 = \\
&= 4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} + 1 = \\
&= 1 + 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.
\end{aligned}$$

Neenakost 4

$$\text{ctg}\alpha \cdot \text{ctg}\beta \cdot \text{ctg}\gamma \leq \frac{\sqrt{3}}{9}$$

Dokaz 4 Če predpostavimo, da eden izmed kotov, recimo α, ni oster. Potem je $\text{ctg}\alpha < 0$. Preostala dva kota sta ostra, zato neenakost očitno velja. Če trikotnik ni topokoten, velja Jansenova neenakost za tangens, saj je to konveksna funkcija na $[0, \pi/2]$:

$$\text{tg}\alpha + \text{tg}\beta + \text{tg}\gamma \geq 3\text{tg}[(\alpha + \beta + \gamma)/3] = 3\sqrt{3}.$$

Ker velja $\text{tg}\alpha \cdot \text{tg}\beta \cdot \text{tg}\gamma = \text{tg}\alpha + \text{tg}\beta + \text{tg}\gamma$ [1], sledi

$$\text{tg}\alpha \cdot \text{tg}\beta \cdot \text{tg}\gamma \geq 3\text{tg}\left((\alpha + \beta + \gamma)/3\right) = 3\sqrt{3} \Rightarrow$$

Z upoštevanjem obratne vrednosti sledi *neenakost 4*. □

Neenakost 5

$$\text{ctg}\alpha + \text{ctg}\beta + \text{ctg}\gamma \geq \sqrt{3}$$

Dokaz 5 Velja

$$\begin{aligned}\text{ctg}\alpha + \text{ctg}\beta &= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\alpha}{\sin\beta} \\ &= \frac{\sin\beta\cos\alpha + \cos\beta\sin\alpha}{\sin\alpha\sin\beta} \\ &= \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.\end{aligned}$$

Upoštevamo

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \leq 1$$

$$-\cos(\alpha + \beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos\gamma.$$

Seštejemo in dobimo

$$2\sin\alpha\sin\beta \leq 1 + \cos\gamma$$

$$2\sin\alpha\sin\beta\sin(\alpha+\beta) \leq (1+\cos\gamma)\sin(\alpha+\beta)$$

$$2\sin\alpha\sin\beta\sin\gamma \leq (1+\cos\gamma)\sin(\alpha+\beta)$$

$$\frac{2\sin\alpha\sin\beta\sin\gamma}{\sin\alpha\sin\beta(1+\cos\gamma)} \leq \frac{(1+\cos\gamma)\sin(\alpha+\beta)}{\sin\alpha\sin\beta(1+\cos\gamma)}$$

$$\frac{2\sin\gamma}{1+\cos\gamma} \leq \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.$$

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[1]

$$0 = \text{tg}(\pi) = \text{tg}(\alpha + \beta + \gamma) = \frac{\text{tg}\alpha + \text{tg}\beta + \text{tg}\gamma - \text{tg}\alpha\text{tg}\beta\text{tg}\gamma}{1 - \text{tg}\alpha\text{tg}\beta - \text{tg}\beta\text{tg}\gamma - \text{tg}\alpha\text{tg}\gamma}$$

Sledi

$$\begin{aligned}
\mathrm{ctg}\alpha+\mathrm{ctg}\beta+\mathrm{ctg}\gamma &= \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}+\mathrm{ctg}\gamma\\
&\geq \frac{2\sin\gamma}{1+\cos\gamma}+\frac{\cos\gamma}{\sin\gamma}\\
&= \frac{1}{2}\left(\frac{4\sin^2\gamma+2\cos^2\gamma+2\cos\gamma}{(1+\cos\gamma)\sin\gamma}\right)\\
&= \frac{1}{2}\left(\frac{3\sin^2\gamma+\cos^2\gamma+2\cos\gamma+1}{(1+\cos\gamma)\sin\gamma}\right)\\
&= \frac{1}{2}\left(\frac{3\sin^2\gamma+(\cos\gamma+1)^2}{(1+\cos\gamma)\sin\gamma}\right)\\
&= \frac{1}{2}\left(\frac{3\sin\gamma}{\cos\gamma+1}+\frac{\cos\gamma+1}{\sin\gamma}\right)\\
\textit{(aritmetično-geometrijska neenakost)}\quad &\geq \frac{2}{2}\left(\sqrt{\frac{3\sin\gamma}{\cos\gamma+1}\frac{\cos\gamma+1}{\sin\gamma}}\right)\\
&= \sqrt{3}
\end{aligned}$$

Torej je $\mathrm{ctg}\alpha+\mathrm{ctg}\beta+\mathrm{ctg}\gamma=\sqrt{3}$. □

Neenakost 6

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma\leq\frac{9}{4}$$

Dokaz 6 Zaradi enakosti sinusa suplementarnih kotov velja

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2(\alpha+\beta).$$

Od tod sledi

$$\begin{aligned}
&\sin^2\alpha+\sin^2\beta+\sin^2\alpha\cos^2\beta+2\sin\alpha\sin\beta\cos\alpha\cos\beta+\cos^2\alpha\sin^2\beta\\
&=\sin^2\alpha+\sin^2\beta+(1-\cos^2\alpha)\cos^2\beta+2\sin\alpha\sin\beta\cos\alpha\cos\beta+\cos^2\alpha(1-\cos^2\beta)\\
&=\sin^2\alpha+\sin^2\beta+\cos^2\beta-\cos^2\alpha\cos^2\beta+2\sin\alpha\sin\beta\cos\alpha\cos\beta+\cos^2\alpha-\cos^2\alpha\cos^2\beta\\
&=2-2\cos^2\alpha\cos^2\beta+2\sin\alpha\sin\beta\cos\alpha\cos\beta\\
&=2-2\cos\alpha\cos\beta(\cos\alpha\cos\beta-\sin\alpha+\sin\beta)\\
&=2-2\cos\alpha\cos\beta\cos(\alpha+\beta)\\
&=2+2\cos\alpha\cos\beta\cos\gamma
\end{aligned}$$

Zaradi (neenakosti 3) velja $\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\leq\frac{1}{2}$. Sledi $\left(\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\right)^3\leq\frac{1}{8}$.
Zaradi geomerijsko-aritmetične neenakosti velja

$$\cos\alpha\cos\beta\cos\gamma\leq\left(\frac{\cos\alpha+\cos\beta+\cos\gamma}{3}\right)^3\leq\frac{1}{8}.$$

Torej je

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2+2\cos\alpha\cos\beta\cos\gamma\leq2+2(\frac{1}{8})=\frac{9}{4}.$$

□

Neenakost 7

$$\mathrm{ctg}^2\alpha+\mathrm{ctg}^2\beta+\mathrm{ctg}^2\gamma\geq1$$

Dokaz 7 Zaradi aritmetično-geometrijske neenakosti je

$$\mathrm{ctg}^2\alpha+\mathrm{ctg}^2\beta\geq2\mathrm{ctg}\alpha\cdot\mathrm{ctg}\beta.$$

Analogno sklepamo za ostala dva para. Če vse tri neenakosti seštejemo in jih delimo z 2, dobimo

$$\begin{aligned}
\mathrm{ctg}^2\alpha + \mathrm{ctg}^2\beta + \mathrm{ctg}^2\gamma &\geq \mathrm{ctg}\beta \cdot \mathrm{ctg}\alpha + \mathrm{ctg}\beta \cdot \mathrm{ctg}\gamma + \mathrm{ctg}\gamma \cdot \mathrm{ctg}\alpha \\
&= \mathrm{ctg}\alpha \cdot \mathrm{ctg}\beta - \mathrm{ctg}\beta \cdot \mathrm{ctg}(\alpha+\beta) - \mathrm{ctg}(\alpha+\beta)\mathrm{ctg}\alpha \\
&= \mathrm{ctg}\alpha \cdot \mathrm{ctg}\beta - \mathrm{ctg}(\alpha+\beta)(\mathrm{ctg}\beta + \mathrm{ctg}\alpha) \\
&= \mathrm{ctg}\alpha \cdot \mathrm{ctg}\beta - \frac{\mathrm{ctg}\alpha \cdot \mathrm{ctg}\beta - 1}{\mathrm{ctg}\alpha + \mathrm{ctg}\beta}(\mathrm{ctg}\beta + \mathrm{ctg}\alpha) \\
&= 1.
\end{aligned}$$

Torej je $\mathrm{ctg}^2\alpha + \mathrm{ctg}^2\beta + \mathrm{ctg}^2\gamma \geq 1$. □

Neenakost 8

$$\sin\alpha \sin\beta \sin\gamma \leq \frac{3\sqrt{3}}{8}$$

Dokaz 8 Zaradi aritmetično-geometrijske neenakosti in neenakosti 1 velja

$$\sqrt[3]{\sin\alpha \sin\beta \sin\gamma} \leq \frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \leq \frac{\sqrt{3}}{2}$$

Potenciramo in odpravimo koren:

$$\sin\alpha \sin\beta \sin\gamma \leq \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8} \tag{1}$$

□

Neenakost 9

$$\csc\alpha \csc\beta \csc\gamma \geq \frac{8\sqrt{3}}{9}$$

Dokaz 9 Po definiciji je $\csc x = \frac{1}{\sin x}$. Po (1) sledi

$$\csc\alpha \csc\beta \csc\gamma = \frac{1}{\sin\alpha \sin\beta \sin\gamma} \geq \left(\frac{3\sqrt{3}}{8}\right)^{-1} = \frac{8\sqrt{3}}{9} \qquad \square$$

Neenakost 10

$$\sec^2\alpha + \sec^2\beta + \sec^2\gamma > 3$$

Dokaz 10 Ker je $|\cos x| \leq 1$, velja $|\sec x| = |\frac{1}{\cos x}| \geq 1,$ za vsak kot x. Sledi

$$\sec^2\alpha + \sec^2\beta + \sec^2\gamma \geq 3.$$

Enačaj ni možen. V primeru enačaja bi veljalo $\alpha = \beta = \gamma = 0°$.
Zato velja

$$\sec^2\alpha + \sec^2\beta + \sec^2\gamma > 3. \qquad \square$$

Neenakost 11

$$\csc^2\alpha + \csc^2\beta + \csc^2\gamma > 4$$

Dokaz 11 Velja zveza $\csc^2\alpha = \dfrac{1}{\sin^2\alpha} = 1 + \mathrm{ctg}^2\alpha$. Analogno velja za kota β in γ. Seštejemo vse tri enakosti in dobimo:

$$\csc^2\alpha + \csc^2\beta + \csc^2\gamma = 1 + \mathrm{ctg}^2\alpha + 1 + \mathrm{ctg}^2\beta + 1 + \mathrm{ctg}^2\gamma.$$

Po neenakosti 7 dobimo

$$\csc^2\alpha + \csc^2\beta + \csc^2\gamma \geq 4. \qquad \square$$

Neenakost 12

$$1 < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2}$$

Dokaz 12 Izhajamo iz enakosti

$$\frac{\pi-\alpha}{2} + \frac{\pi-\beta}{2} + \frac{\pi-\gamma}{2} = \pi. \tag{2}$$

Po neenakosti 3 sledi

$$1 < \cos\left(\frac{\pi-\alpha}{2}\right) + \cos\left(\frac{\pi-\beta}{2}\right) + \cos\left(\frac{\pi-\gamma}{2}\right) \le \frac{3}{2}.$$

Upoštevamo $\sin x = \cos(\pi/2 - x)$, od tu sledi

$$1 < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2}. \qquad \square$$

Neenakost 13

$$\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{2}$$

Dokaz 13 Tudi tu podobno kot v prejšnjem primeru uporabimo enakost (2). Po neenakosti 1 sledi

$$\sin\left(\frac{\pi-\alpha}{2}\right) + \sin\left(\frac{\pi-\beta}{2}\right) + \sin\left(\frac{\pi-\gamma}{2}\right) \le \frac{3\sqrt{3}}{2}$$

$$\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{2}. \qquad \square$$

Neenakost 14

$$\mathrm{tg}\frac{\alpha}{2} + \mathrm{tg}\frac{\beta}{2} + \mathrm{tg}\frac{\gamma}{2} \ge \sqrt{3}$$

Dokaz 14 Po (2), neenakosti 5 in zvezi $\mathrm{ctg}(\pi/2 - x) = \mathrm{tg}x$ je

$$\mathrm{ctg}\left(\frac{\pi-\alpha}{2}\right) + \mathrm{ctg}\left(\frac{\pi-\beta}{2}\right) + \mathrm{ctg}\left(\frac{\pi-\gamma}{2}\right) \ge \sqrt{3}$$

$$\mathrm{tg}\frac{\alpha}{2} + \mathrm{ctg}\frac{\beta}{2} + \mathrm{ctg}\frac{\gamma}{2} \ge \sqrt{3} \qquad \square$$

Neenakost 15

$$\mathrm{ctg}\frac{\alpha}{2} + \mathrm{ctg}\frac{\beta}{2} + \mathrm{ctg}\frac{\gamma}{2} \ge 3\sqrt{3}$$

Dokaz 15 Po (2) je $\frac{\pi-\alpha}{2} + \frac{\pi-\beta}{2} + \frac{\pi-\gamma}{2} = \pi$. Vsak izmed sumandov na levi strani je oster kot v trikotniku, tangens je konveksna funkcija na $[0, \pi/2)$, zato po Jansenovi neenakosti velja

$$\frac{\mathrm{tg}(\frac{\pi-\alpha}{2}) + \mathrm{tg}(\frac{\pi-\beta}{2}) + \mathrm{tg}(\frac{\pi-\gamma}{2})}{3} \ge \mathrm{tg}\left(\frac{\frac{\pi-\alpha}{2} + \frac{\pi-\beta}{2} + \frac{\pi-\gamma}{2}}{3}\right) = \mathrm{tg}\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Ker velja $\mathrm{ctg}x = \mathrm{tg}(\pi/2 - x)$ sledi

$$\mathrm{ctg}\frac{\alpha}{2} + \mathrm{ctg}\frac{\beta}{2} + \mathrm{ctg}\frac{\gamma}{2} \ge 3\sqrt{3}. \qquad \square$$

Neenakost 16

$$\csc\frac{\alpha}{2} + \csc\frac{\beta}{2} + \csc\frac{\gamma}{2} \geq 6$$

Dokaz 16 Funkcija $\csc x$ je konveksna na $(0, \pi/2]$,zato je $f(x) = \csc(x/2)$ konveksna na intervalu $[0, \pi]$. Za f zapišemo Jansenovo neenakost in dobimo:

$$\frac{\csc(\alpha/2) + \csc(\beta/2) + \csc(\gamma/2)}{3} \geq \csc\left(\frac{\alpha/2 + \beta/2 + \gamma/2}{3}\right) = \frac{1}{\sin(\pi/6)} = 2$$

Sledi

$$\csc\left(\frac{\alpha}{2}\right) + \csc\left(\frac{\beta}{2}\right) + \csc\left(\frac{\gamma}{2}\right) \geq 6. \qquad \square$$

Neenakost 17

$$\sec\frac{\alpha}{2} + \sec\frac{\beta}{2} + \sec\frac{\gamma}{2} \geq 2\sqrt{3}$$

Dokaz 17 S podobnim razmislekom kot prej je funkcija $\sec(x/2)$ konveksna na $[0, \pi]$. Sledi

$$\frac{\sec(\alpha/2) + \sec(\beta/2) + \sec(\gamma/2)}{3} \geq \sec\left(\frac{\alpha/2 + \beta/2 + \gamma/2}{3}\right) = \frac{1}{\cos(\pi/6)} = \frac{2}{\sqrt{3}}$$

Sledi

$$\sec\frac{\alpha}{2} + \sec\frac{\beta}{2} + \sec\frac{\gamma}{2} \geq 2\sqrt{3}. \qquad \square$$

Literatura

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